

Thm 1.3 (Chinese Remainder Theorem, CRT)

If $I_1, \dots, I_n \triangleleft A$ are pairwise comaximal (i.e. $I_i + I_j = A$ for $i \neq j$),

$$\text{then } A / \bigcap_{i=1}^n I_i \cong A / I_1 \times \dots \times A / I_n,$$

$$\text{and } I_1 \dots I_n = I_1 \cap \dots \cap I_n$$

$$\text{Exm: } \mathbb{Z} / p_1^n \mathbb{Z} \cong \mathbb{Z} / p_1^n \mathbb{Z} \times \mathbb{Z} / p_2^n \mathbb{Z} \text{ for } p \neq q \text{ primes.}$$

Proof of CRT skipped / Exercise

1.2 Modules

Let A be a ring.

Module M : $(M, +)$ abelian group, with \circ scalar multiplication

$$A \times M \rightarrow M, (a, m) \mapsto am \text{ s.t.}$$

$$1_A m = m, (ab)m = a(bm), a(m+n) = am + an,$$

$$(a+b)m = am + bm \quad (\forall a, b \in A, m, n \in M)$$

If M is an abelian group, an A -module structure can equivalently be described by a ring hom $E: A \rightarrow \text{End}(M)$ (structure hom.)

$$[\text{"} \Rightarrow \text{" } E(a) := \mu_a \text{ where } \mu_a(m) = am$$

$$\text{"} \Leftarrow \text{" Given } E: A \rightarrow \text{End}(M), \text{ define } a \cdot m := E(a)(m)]$$

Exm: \cdot) \mathbb{Z} -modules $\hat{=}$ abelian groups

\cdot) K field, K -modules $\hat{=}$ K -vector spaces

\cdot) Submodules of A are the ideals of A

\cdot) If V K -vector space, $\varphi \in \text{End}(V_K)$, then

$$\Phi: K[x] \rightarrow \text{End}(V), \sum_{i=0}^n a_i x^i \mapsto \sum_{i=0}^n a_i \varphi^i \text{ is a ring hom.}$$

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Induces $K[x]$ -module structure on V , with x acting as φ :

$$x \cdot v = \varphi(v).$$

$(\text{Ker}(\Phi) = (m_\varphi))$ with m_φ minimal polynomial of φ

M A -module, $E \subseteq M \Rightarrow \langle E \rangle_A = \left\{ \sum_{i=1}^n a_i m_i \mid a_i \in A, m_i \in E \right\}$ is

the A -module **generated by E** .

If $(N_i)_{i \in I}$, $N_i \subseteq M$ ^{submodule} $\Rightarrow \sum_{i \in I} N_i = \langle \bigcup_{i \in I} N_i \rangle$, $\bigcap_{i \in I} N_i$ are

submodules.

If $I \trianglelefteq A$: $IM = \left\{ \sum_{i=1}^n a_i m_i \mid a_i \in I, m_i \in M \right\} \subseteq M$

Remark: Let $\varphi: A \rightarrow B$ be a ring hom.

(1) If N_B B -module $\Rightarrow N$ is an A -module via $a \cdot n := \varphi(a)n$.

(2) If φ is an epic, $I = \ker(\varphi)$, M_A A -module and $IM = 0$, then M is a B -module via

$$b \cdot m := \varphi^{-1}(b)m \text{ for any } a \in \varphi^{-1}(b)$$

In particular: $M \in A\text{-Mod}$, $I \trianglelefteq A \Rightarrow M/IM$ is A/I -module

$$\text{with } (a+I)(m+IM) = am + IM.$$

Category equivalence: $\{A\text{-modules } M \text{ with } IM=0\} \leftrightarrow A/I\text{-Mod.}$

Exm: M abelian group (\mathbb{Z} -module), p prime

$\Rightarrow M/pM$ is $\mathbb{Z}/p\mathbb{Z}$ vector space

Homomorphisms $f: M_A \rightarrow N_A$ hom. $\Rightarrow \ker(f) \subseteq M_A$, $f(M) \subseteq N$

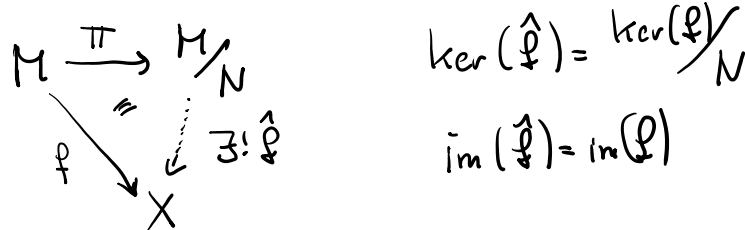
Quotients: $N_A \subseteq M_A \Rightarrow M/N = \{m+N \mid m \in M\}$ is A -module

$$\text{with } a(m+N) = (am)+N$$

with $a(m+N) = (am)+N$

Canonical epi $\pi: M \rightarrow M/N, m \mapsto m+N$

Universal Property of $(M/N, \pi)$: If $f: M_A \rightarrow X_A$ is a hom. with $N \subseteq \ker(f)$, then there exists a unique hom. $\hat{f}: M/N \rightarrow X$ s.t. $f = \hat{f} \circ \pi$

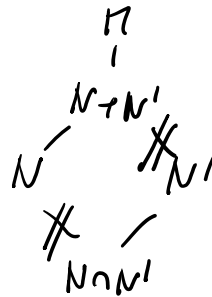


Isomorphism Theorems: $f(M) \cong \frac{M}{\ker(f)}$

$\{X_A, N \subseteq X \subseteq M\} \xleftrightarrow{\text{bij}} \{\text{submodules of } M/N\}$

$N \subseteq X \subseteq M, \frac{M}{X} \cong \frac{(M/N)}{(X/N)}$

$N, N' \subseteq M: \frac{N+N'}{N} \cong \frac{N'}{N \cap N'}$



More Constructions

$(M_i)_{i \in I}$ family of A -modules

Product

$\prod_{i \in I} M_i = \{ (m_i)_i : m_i \in M_i \}$

Direct Sum (Coproduct)

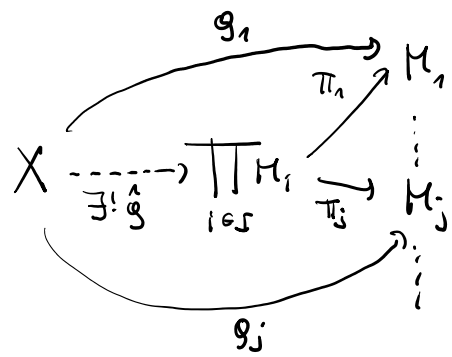
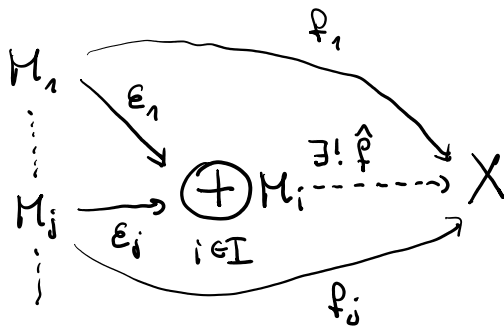
$\bigoplus_{i \in I} M_i = \{ (m_i)_i : m_i \in M_i, m_i = 0 \text{ for finitely many } i \}$

Hom's:

$\epsilon_j: \begin{cases} M_j \rightarrow \bigoplus_{i \in I} M_i \\ m \mapsto (m_i)_i \text{ with } m_i = \begin{cases} 0 & \text{if } i \neq j \\ m & \text{if } i = j \end{cases} \end{cases}$

$\text{little } \pi \downarrow \pi_j: \begin{cases} \prod_{i \in I} M_i \rightarrow M_j \\ (m_i)_{i \in I} \mapsto m_j \end{cases}$ (bis π (product))

UP₃:



(UP \oplus): For every family $(f_i: M_i \rightarrow X)_i$ of A -hom's, there exists a unique A -hom $\hat{f}: \bigoplus_{i \in I} M_i \rightarrow X$ s.t. $\forall j: \hat{f} \circ e_j = f_j$

(UP \prod): For every family $(g_i: X \rightarrow M_i)_i$ of A -hom's, there exists a unique A -hom $\hat{g}: X \rightarrow \prod_{i \in I} M_i$ s.t. $\forall j: \pi_j \circ \hat{g} = g_j$.