W1/2 Modules (Recap)

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Thm 1.3 (Chinese Remainder Theorem, CRT)

If $I_1, \dots, I_n \trianglelefteq A$ are pairwise comaximal (i.e. $I_i + I_j = A$ for $i \neq j$),

then $$A/I_1 \cap \dots \cap I_n \cong A/I_1 \times \dots \times A/I_n,$$

and $I_1 \cap \dots \cap I_n = I_1 \cdots I_n$

Exm: $\mathbb{Z}/p^n q^n \mathbb{Z} \cong \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/q^n\mathbb{Z}$ for $p \neq q$ primes.

Proof of CRT skipped / Exercise

1.2 Modules

Let A be a ring.

Module M: $(M,+)$ abelian group, with a scalar multiplication

$A \times M \to M$, $(a,m) \mapsto am$ s.t.

$1_A m = m$, $(ab)m = a(bm)$, $a(m+n) = am + an$,

$(a+b)m = am + bm$ $\quad (\forall a,b \in A,\ m,n \in M)$

If M is an abelian group, an A-module structure can equivalently be described by a ring hom $\mathcal{E}: A \to End(M)$ (structure hom.)

[„$\Rightarrow$" $\mathcal{E}(a) := \mu_a$ where $\mu_a(m) = am$

„$\Leftarrow$" Given $\mathcal{E}: A \to End(M)$, define $a \cdot m := \mathcal{E}(a)(m)$]

Exm:
- $\mathbb{Z}$-modules $\hat{=}$ abelian groups
- K field, K-modules $\hat{=}$ K-vector spaces
- Submodules of A are the ideals of A
- If V K-vector space, $\varphi \in End(V_K)$, then

$$\Phi: K[x] \to End(V),\ \sum_{i=0}^{n} a_i x^i \mapsto \sum_{i=0}^{n} a_i \varphi^i$$ is a ring hom.

$\Phi: K[x] \longrightarrow \mathrm{End}(V)$, $\sum_{i=1} a_i x^i \mapsto \sum_{i=1} a_i \varphi^i$ is a ring hom.

Induces $K[x]$-module structure on V, with x acting as φ:

$x \cdot v = \varphi(v)$.

($\mathrm{Ker}(\Phi) = (m_\varphi)$ with m_φ minimal polynomial of φ)

M A-module, $E \subseteq M \Rightarrow \langle E \rangle_A = \left\{ \sum_{i=1}^{n} a_i m_i \mid a_i \in A, m_i \in E \right\}$ is the A-module **generated by E**.

If $(N_i)_{i \in I}$, $N_i \overset{\text{submodule}}{\leqslant} M \Rightarrow \sum_{i \in I} N_i := \langle \bigcup_{i \in I} N_i \rangle$, $\bigcap_{i \in I} N_i$ are submodules.

If $I \trianglelefteq A$: $IM = \left\{ \sum_{i=1}^{n} a_i m_i : a_i \in I, m_i \in M \right\} \leqslant M$

<u>Remark:</u> Let $\varphi: A \to B$ be a ring hom.

(1) If N_B B-module $\Rightarrow N$ is an A-module via $a \cdot n := \varphi(a) n$.

(2) If φ is an <u>epi</u>, $I := \ker(\varphi)$, M_A A-module and $IM = 0$, then M is a B-module via

$b \cdot m := am$ for <u>any</u> $a \in \varphi^{-1}(b)$

<u>In particular:</u> $M \in A\text{-Mod}$, $I \trianglelefteq A \to M/_{IM}$ is $A/_I$-module

with $(a+I)(m+IM) = am + IM$.

Category equivalence: $\{A\text{-modules } M \text{ with } IM = 0\} \leftrightarrow A/_I\text{-Mod}$.

<u>Exm:</u> M abelian group ($\mathbb{Z}$-module), p prime

$\Rightarrow M/_{pM}$ is $\mathbb{Z}/_{p\mathbb{Z}}$ vector space

<u>Homomorphisms</u> $f: M_A \to N_A$ hom. $\Rightarrow \ker(f) \leqslant M_A$, $f(M) \leqslant N$

<u>Quotients:</u> $N_A \leqslant M_A \to M/_N = \{m + N : m \in M\}$ is A-module

with $a(m+N) = (am) + N$

with $a(m+N) = (am)+N$

Canonical epi $\pi : M \to M/N$, $m \mapsto m+N$

Universal Property of $(M/N, \pi)$: If $f : M_A \to X_A$ is a hom. with $N \subseteq \ker(f)$, then there exists a unique hom. $\hat{f} : M/N \to X$ s.t. $f = \hat{f} \circ \pi$

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[dr, "f"'] & M/N \arrow[d, dashed, "\exists!\,\hat{f}"] \\ & X \end{tikzcd}$$

$\ker(\hat{f}) = \ker(f)/N$

$\operatorname{im}(\hat{f}) = \operatorname{im}(f)$

Isomorphism Theorems: $f(M) \cong M/\ker(f)$

$\{X_A : N \leq X \leq M\} \overset{\text{bij}}{\longleftrightarrow} \{\text{submodules of } M/N\}$

$N \leq X \leq M$: $M/X \cong (M/N)/(X/N)$

$N, N' \leq M$: $(N+N')/N \cong N'/(N \cap N')$

$$\begin{tikzcd}[column sep=small] & M \arrow[d, no head] & \\ & N+N' \arrow[dl, no head] \arrow[dr, no head] & \\ N \arrow[dr, no head] & & N' \arrow[dl, no head] \\ & N \cap N' & \end{tikzcd}$$

More Constructions

$(M_i)_{i \in I}$ family of A-modules

Product $\prod_{i \in I} M_i = \{(m_i)_i : m_i \in M_i\}$

Direct Sum (Coproduct) $\bigoplus_{i \in I} M_i = \{(m_i)_i : m_i \in M_i,\ m_i \neq 0$ for finitely many $i\}$

$\exists$ hom's:

$$\varepsilon_j : \begin{cases} M_j \to \bigoplus_{i \in I} M_i \\ m \mapsto (m_i)_i \text{ with } m_i = \begin{cases} 0 & \text{if } i \neq j \\ m & \text{if } i = j \end{cases} \end{cases}$$

big Π (product)

little π

$$\pi_j : \begin{cases} \prod_{i \in I} M_i \to M_j \\ (m_i)_{i \in I} \mapsto m_j \end{cases}$$

UPs:

$$\begin{tikzcd}
M_1 \arrow[dr, "\varepsilon_1"] \arrow[drr, bend left, "f_1"] & & \\
\vdots & \bigoplus_{i\in I} M_i \arrow[r, dashed, "\exists!\,\hat{f}"] & X \\
M_j \arrow[ur, "\varepsilon_j"'] \arrow[urr, bend right, "f_j"'] & & \\
\vdots & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & M_1 \\
X \arrow[r, dashed, "\exists!\,\hat{g}"'] \arrow[urr, bend left, "g_1"] \arrow[drr, bend right, "g_j"'] & \prod_{i\in I} M_i \arrow[ur, "\pi_1"] \arrow[dr, "\pi_j"'] & \vdots \\
 & & M_j \\
 & & \vdots
\end{tikzcd}$$

(UP$\oplus$): For every family $(f_i : M_i \to X)_i$ of A-hom's, there exist a unique A-hom $\hat{f} : \bigoplus_{i\in I} M_i \to X$ s.t. $\forall j : \hat{f} \circ \varepsilon_j = f_j$

(UP$\prod$): For every family $(g_i : X \to M_i)_i$ of A-hom's, there exist a unique A-hom $\hat{g} : X \to \prod_{i\in I} M_i$ s.t. $\forall j : \pi_j \circ \hat{g} = g_j$.